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ON A METHOD OF SOLVING A TWO-DIMENSIONAL INTEGRAL EQUATION OF THE FIRST KIND WITH A POWER-LAW KERNEL AND ITS APPLICATION TO CONTACT PROBLEMS*

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A method based on Hobson's theorem /1/ is proposed for constructing exact solutions of an integral equation of the first kind, defined on an elliptical area, with a power-law (polar) kernel and a polynomial right side. Solutions of this equation with different asymptotic expansions in the neighbourhood of the boundary ellipse are presented in an explicit closed form. The problem of the pressure of a stiff elliptical cylinder with arbitrary polynomial form for the base in an inhomogeneous elastic half-space ($\nu = \text{const}, E = E_\alpha x_3^\alpha$) is considered as an illustration.

Rostovtsev /2,3/ earlier indicated just the functional form of the unbounded solution of the equation mentioned, but did not obtain a relationship to define the constant coefficients in this solution in closed form. The solution for the case when the right side of the integral equation is a polynomial of zeroth power is given in /4/. Results of an investigation of integral equations of the first kind with power-law kernel defined on circular and strip areas are presented in /4,5/.

Utilization of Hobson's theorem and the linear recurrence relations obtained below for the generalized potential factors of an elliptical disc, enables us, in addition to the rest, to get rid completely of the awkward apparatus introduced in /3/ for functions generated by Lamé ellipsoidal functions.

1. We denote points of the real Euclidean space R^3 by $\mathbf{x} = (x_1, x_2, x_3)$ and the points of R^2 by $\mathbf{x}_0 = (x_1, x_2)$ and $\mathbf{y}_0 = (y_1, y_2)$. Let $R^2 \supset \Omega = \{\mathbf{x}_0: x_1^2/a_1^2 + x_2^2/a_2^2 \leqslant 1\}$. We examine the following two-dimensional integral equation of the first kind with power-law kernel in the function $p(\mathbf{x}_0)$:

$$\gamma \mathbf{U}_\xi p(\mathbf{x}_0) = q(\mathbf{x}_0), \quad \mathbf{x}_0 \in \Omega \tag{1.1}$$

$$\mathbf{U}_\xi p(\mathbf{x}) = \iint_\Omega |\mathbf{x} - \mathbf{y}_0|^{-\xi} p(\mathbf{y}_0)\, d\mathbf{y}_0$$

where the function q and the constant γ are assumed known, while the parameter ξ takes arbitrary values in the interval (0,2). Boundary value problems for equations of elliptic type reduce to integral equations with such operators.

The integral operator $\mathbf{U}_\xi$ is a generalized potential (or Reisz potential) with density p distributed over an elliptical disc Ω. The properties of the operators $\mathbf{U}_\xi$ acting in spaces of summable functions were discussed in /6/.

The values of the integral $\mathbf{U}_\xi p(\mathbf{x})$ at the points $\mathbf{x}_0 \in \Omega$ determine the internal generalized potential of an elliptical disc which we denote by $\mathbf{U}_\xi^\Omega p(\mathbf{x}_0)$.

Furthermore, we shall assume that

$$q(\mathbf{x}_0) = Q_l(\mathbf{x}_0) \equiv \sum_{m+n=0}^{l} q_{mn} x_1^m x_2^n \quad (q_{mn} = \text{const}) \tag{1.2}$$

where l is an arbitrary non-negative integer. Such an assumption is fairly general since any function $q \in C^k(\Omega)$, say, can be approximated uniformly by using polynomials so that the derivatives of q to k-th order would be approximated uniformly by derivatives of these polynomials of corresponding order /7/.

The integral equation (1.1) is here written as follows:

$$\mathbf{U}_\xi^\Omega p(\mathbf{x}_0) = \gamma^{-1} Q_l(\mathbf{x}_0) \tag{1.3}$$

Its solution is equivalent to that selection of the density p for which the internal generalized potential of an elliptical disc would become the given polynomal $\gamma^{-1}Q_l$. The

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main result referring to the generalized potentials of an elliptical disc was obtained by Hobson /1/. A theorem he established enables a double integral of $U_\xi p(\mathbf{x})$ to be reduced to a sum (infinite in the general case) of single integrals.

Using this theorem and omitting the awkward intermediate calculations, we can show that if

$$p^{(k)}(\mathbf{x}_0) = \left(1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right)^{k+\xi/2-1} \sum_{m+n=0}^{l-2k} p_{mn}^{(k)} x_1{}^m x_2{}^n \tag{1.4}$$

where $p_{mn}^{(k)}$ are arbitrary constants $(k = 0, 1, \ldots, [l/2])$, then

$$U_\xi^\Omega p(\mathbf{x}_0; k) = \sum_{m+n=0}^{l} u_{mn}^{(k)}(p_{vw}^{(k)}) x_1{}^m x_2{}^n \tag{1.5}$$

$$u_{mn}^{(k)}(p_{vw}^{(k)}) = \frac{(\xi/2)_k}{k!} \sum_{r=0}^{[l/2]-k} \frac{2^{1-2r}}{r!\,(k+1)_r} \times \sum_{\substack{i=0\\(i,\,j\in J)}}^{[m/2]} \sum_{j=0}^{[n/2]} (-1)^{i+j} C_{k+r}^i C_{k+r-i}^j \sum_{s=0}^{r} C_r{}^s (m-2i+1)_{2r-2s} \times$$
$$(n-2j+1)_{2s} a_1^{2(M-i)} a_2^{2(N-j)} B_{M,\,N,\,r}^{(\xi)} p_{2M-m,\,2N-n}^{(k)}$$

Here $[l/2]$ is the integer part of the number $l/2$, $(m)_r$ is the Pochhammer symbol, $C_k{}^i$ is the binomial coefficient, $M = m - i + r - s$, $N = n - j + s$, $J = \{i, j\colon k + r - {}^1/_2\,(l - m - n) \leqslant i + j \leqslant k + r\}$, and also

$$B_{m,\,n,\,r}^{(\xi)} = \frac{\pi a_1 a_2}{2} \int_0^\infty \frac{\theta^{r-\xi/2}\, d\theta}{(a_1^2+\theta)^{m+1/2}(a_2^2+\theta)^{n+1/2}} \tag{1.6}$$

Thus, if the density of the generalized potential has the form (1.4), then the integral equation (1.3) reduces to the following system of linear algebraic equations in the constants $p_{vw}^{(k)}$:

$$u_{mn}^{(k)}(p_{vw}^{(k)}) = \gamma^{-1} q_{mn} \quad (m + n = 0, 1, \ldots, l) \tag{1.7}$$

The system (1.7) consists of $t = {}^1/_2\,(l+1)(l+2)$ equations and contains $t_k = {}^1/_2\,(l - 2k + 1)(l - 2k + 2)$ unknowns so that $t \geqslant t_k$ and the equality sign holds just for $k = 0$. Consequently, the solution of this system, and therefore, of the integral equations (1.3) also can be constructed for arbitrary values of the constants q_{mn} only when $k = 0$. For the existence of solutions of the type (1.4) for $k > 0$ it is necessary (and sufficient) that the constants q_{mn} satisfy a set of relationships resulting from (1.7) whose quantity equals $t - t_k$.

All possible solutions of the integral equation (1.3) are determined by the functions $p^{(k)}(\mathbf{x}_0)$ $(k = 0, 1, \ldots, [l/2])$. Rostovtsev /2,3/ earlier indicated just the functional form of the solution $p^{(0)}(\mathbf{x}_0)$ but did not obtain a system of algebraic equations in the constants $p_{vw}^{(0)}$ in closed form.

2. In finding the coefficients of system (1.7) it is necessary to evaluate the improper single integrals $B_{m,\,n,\,r}^{(\xi)}$ which we shall call the generalized internal potential factors of an elliptical disc. The most logical method for this calculation is to use a reduction formula.

The validity of the following recursion relations is established by direct substitution

$$B_{m,\,n,\,r}^{(\xi)} = B_{m-1,\,n,\,r-1}^{(\xi)} - a_1^2 B_{m,\,n,\,r-1}^{(\xi)} \tag{2.1}$$
$$B_{m,\,n,\,r}^{(\xi)} = B_{m,\,n-1,\,r-1}^{(\xi)} - a_2^2 B_{m,\,n,\,r-1}^{(\xi)}$$

which will enable us to represent the arbitrary factor $B_{m,\,n,\,r}^{(\xi)}$ in the form of a linear combination of the factors $B_{v,\,w,\,0}^{(\xi)} \equiv B_{v,\,w}^{(\xi)}$.

The relationship

$$(a_1^2 - a_2^2) B_{m,\,n}^{(\xi)} = B_{m-1,\,n}^{(\xi)} - B_{m,\,n-1}^{(\xi)} \tag{2.2}$$

that holds for $m \neq 0$ and $n \neq 0$ is also confirmed directly.

Utilization of the Euler formula for homogeneous functions results in an important auxiliary result

$$(2m+1)\, a_1^2 B_{m+1,n}^{(\xi)} + (2n+1)\, a_2^2 B_{m,n+1}^{(\xi)} = (2m+2n+\xi)\, B_{m,}^{(\xi)} \tag{2.3}$$

The relationships

$$(2m+3)\, a_1^2 (a_1^2 - a_2^2)\, B_{m+2,\,0}^{(\xi)} = -(2m+\xi)\, B_{m,\,0}^{(\xi)} + \tag{2.4}$$
$$[a_1^2(4m+\xi+3) - a_2^2(2m+\xi+1)]\, B_{m+1,\,0}^{(\xi)}$$

$$(2n+3)a_2^2(a_1^2-a_2^2)B_{0,n+2}^{(\xi)}=(2n+\xi)B_{0,n}^{(\xi)}+$$
$$[a_1^2(2n+\xi+1)-a_2^2(4n+\xi+3)]B_{0,n+1}^{(\xi)}$$

completing the system of reduction formulas, result from (2.2) and (2.3).

The recurrence relations (2.1),(2.2) and (2.4) enable us to represent an arbitrary internal potential factor in the form of a linear combination of three main factors $B_{1,0}^{(\xi)}$, $B_{0,1}^{(\xi)}$, and $B_{0,0}^{(\xi)}$ which are not, however, independent. We obtain the equation relating them from (2.3) for $m=n=0$. Consequently, for an efficient evaluation of the integrals $B_{m,n,r}^{(\xi)}$ it is sufficient to compile tables of values of any two out of the three main factors for different ratios a_2/a_1.

For $\xi=1$ when the generalized potential goes over into the harmonic potential of a simple layer, the internal potential factors of an elliptical disc result in complete elliptic integrals of the first and second kinds /8/.

3. The selection of the kind of solution of the integral equation (1.3) in contact problems is often determined by an a priori knowledge of the nature of the asymptotic behaviour of the solution in the neighbourhood of the boundary ellipse $\Gamma=\{\mathbf{x}_0\colon x_1^2/a_1^2+x_2^2/a_2^2=1\}$, or in parametric form $\Gamma=\{\mathbf{x}_0\colon x_1=a_1\cos\varphi,\ x_2=a_2\sin\varphi\}$, where φ is the parametric angle of the ellipse.

The solution $p^{(0)}(\mathbf{x}_0)$ evidently has a singularity at points of the boundary ellipse, while the solutions $p^{(k)}(\mathbf{x}_0)$ vanish at these points for $k>0$. A more detailed investigation of the asymptotic behaviour of the solutions of the integral equation (1.3) is of interest.

For the points $\mathbf{x}_0\in\Omega$ the following representations hold /9/:

$$x_1=(a_1-\rho\varkappa_1\Psi^{-1})\cos\varphi,\quad x_2=(a_2-\rho\Psi^{-1})\sin\varphi \tag{3.1}$$
$$\Psi=(1-\varkappa^2\cos^2\varphi)^{1/2},\ \varkappa_1=a_2/a_1,\ \varkappa^2=1-\varkappa_1^2$$

where is the distance to Substituting (3.1) into (1.4), we obtain

$$p^{(k)}(\rho)=(2\rho)^{k+\xi/2-1}L_k(\varphi)+O(\rho^{k+\xi/2}),\ \rho\to 0 \tag{3.2}$$

The function $L_k(\varphi)$ is the coefficient of the principal term in the asymptotic expansion of the solution of the integral equation (1.3) and governs the local behaviour of the solution $p^{(k)}(\mathbf{x}_0)$ in the neighbourhood of Γ. We have

$$L_k(\varphi)=\lim_{\rho\to0}(2\rho)^{1-k-\xi/2}p^{(k)}(\rho)=\left(\frac{\Psi}{a_2}\right)^{k+\xi/2-1}\sum_{m+n=0}^{l-2k}p_{mn}^{(k)}(a_1\cos\varphi)^m(a_2\sin\varphi)^n \tag{3.3}$$

We note that the function Ψ lends itself to two simple geometric interpretations, namely

$$\Psi=a_1^{-2/3}a_2^{1/3}R^{1/3}=a_2^{-1}n$$

where R is the radius of curvature, and n is the length of a section of the normal for the ellipse Γ.

4. As an illustration, we examine the problem of the pressure (when there is no friction) of a stiff elliptic cylinder with semi-axes a_1 and a_2 on an inhomogeneous, isotropic half-space $x_3\geqslant0$, whose Poisson's ratio ν is constant while the elastic modulus varies with depth as given by the power law $E=E_\alpha x_3^\alpha$ $(0\leqslant\alpha<1)$. An elastic half-space with such characteristics is sometimes called a quasiclassical foundation /5/.

We take the function describing the shape of the stamp base in the form

$$f(\mathbf{x}_0)=Q_l^\circ(\mathbf{x}_0)\equiv\sum_{m+n=2}^{l}\mathring{q}_{mn}x_1^mx_2^n,\quad \mathbf{x}_0\in\Omega \tag{4.1}$$

Determination of the contact pressure $p=-\sigma_{33}$ here reduces to solving the integral equation (1.3) in which $\xi=\alpha+1$ should be set (see /5,10/) and

$$\gamma=\frac{D\Gamma(1/2+\alpha/2)}{2\pi^{1/2}\Gamma(1+\alpha/2)},\quad D=\frac{(1-\nu^2)C\eta}{\pi(1+\alpha)E_\alpha\Gamma(\alpha+2)}\sin\frac{\pi\eta}{2}$$
$$C=2^{1+\alpha}\Gamma(1/2(\alpha+\eta+3))\,\Gamma(1/2(\alpha-\eta+3))$$
$$\eta^2=1+\alpha-\alpha(1+\alpha)\nu(1-\nu)^{-1},\ q_{0,0}=\delta$$
$$q_{1,0}=-\beta_2,\ q_{0,1}=\beta_1,\ q_{mn}=-q_{mn}^\circ\ (m+n>1)$$

The constants δ,β_1,β_2, not known in advance, denote the translational displacement and rotation vector projection of the stamp, and $\Gamma(\alpha)$ is the Gamma function.

The principal vector and principal moments of the forces applied to the stamp have the form

$$P=\iint_\Omega p\,d\mathbf{x}_0,\quad M_1=\iint_\Omega x_2p\,d\mathbf{x}_0,\quad M_2=-\iint_\Omega x_1p\,d\mathbf{x}_0 \tag{4.2}$$

and are assumed known.

We select

$$p^{(0)}(\mathbf{x}_0) = \left(1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right)^{1/2(\alpha-1)} \sum_{m+n=0}^{l} p_{mn}^{(0)} x_{1m} x_2^n \tag{4.3}$$

as the solution of the integral equation (1.3) that has a singularity at the points $\mathbf{x}_0 \in \Gamma$, which results in the following set of linear algebraic equations in δ, β_1, β_2 and $p_{vw}^{(0)}$:

$$u_{mn}^{(0)}(p_{vw}^{(0)}) = \gamma^{-1} q_{mn} \ (m + n = 0, 1, \ldots, l) \tag{4.4}$$

The stamp equilibrium conditions (4.2) yield three other linear algebraic equations in $p_{vw}^{(0)}$

$$P = \sum_{m+n=0}^{[l/2]} t_{2m,2n} p_{2m,2n}^{(0)}, \quad M_1 = \sum_{m+n=0}^{[1/2(l-1)]} t_{2m,2n+2} p_{2m,2n+1}^{(0)} \tag{4.5}$$

$$M_2 = -\sum_{m+n=0}^{[1/2(l-1)]} t_{2m+2,2n} p_{2m+1,2n}^{(0)}$$

$$t_{2m,2n} = \frac{\pi a_1^{2m+1} a_2^{2n+1} (2m-1)!!\,(2n-1)!!}{2^{m+n} (\xi/2)_{m+n+1}}$$

Therefore, if a stiff elliptical cylinder whose base surface is described by (4.1) is impressed in a quasiclassical foundation, the contact pressure has the form (4.3), while the constants $p_{vw}^{(0)}$ and the parameters δ, β_1, β_2 are found from the set of linear algebraic equations (4.4) and (4.5). The result formulated generalizes Rostovtsev's result /2,3/.

The solution of the appropriate contact problem for a homogeneous half-space with elastic modulus E is obtained as a special case for $\alpha = 0$ when $\xi = 1$ and $\gamma = (1 - \nu^2)(\pi E)^{-1}$.

It should be noted that the solution of the appropriate contact problem of non-linear steady creep with power-law coupling between the stress intensities and the strain rates, obtained within the framework of the principle of superposing "generalized" displacements /11/ (some constraints on such an approach are mentioned in /12/) also results in an integral equation of type (1.3).

5. Let a stiff elliptic cylinder with a flat base be subjected to a central force P in an inhomogeneous half-space. In this case, on the basis of the results in Sec. 4, we have for the stamp contact pressure and settling

$$p^{(0)}(\mathbf{x}_0) = \frac{(1+\alpha)P}{2\pi a_1 a_2} \left(1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right)^{1/2(\alpha-1)} \tag{5.1}$$

$$\delta = (1 + \alpha)\gamma P (\pi a_1 a_2)^{-1} B_{0,0}^{(\xi)}$$

which agrees with the solution obtained in /4/ by another method.

We also consider the problem of the central impression of a stiff elliptic cylinder for which the base shape is described by the function

$$f(\mathbf{x}_0) = \mathring{q}_{2,0} x_1^2 + \mathring{q}_{0,2} x_2^2, \quad \mathbf{x}_0 \in \Omega \tag{5.2}$$

in an inhomogeneous half-space.

Using the results in Sec. 4, we find that the contact pressure in this case is given by the formula

$$p^{(0)}(\mathbf{x}_0) = (p_{0,0}^{(0)} + p_{2,0}^{(0)} x_1^2 + p_{0,2}^{(0)} x_2^2) \left(1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right)^{1/2(\alpha-1)} \tag{5.3}$$

while the constants in (5.3) and the stamp settling satisfy the relationships

$$(2a_1^4 B_{2,0,0}^{(\xi)} - a_1^2 B_{2,0,1}^{(\xi)}) p_{2,0}^{(0)} - a_2^2 B_{1,1,1}^{(\xi)} p_{0,2}^{(0)} = -\gamma^{-1} q_{2,0}^0 \tag{5.4}$$

$$(2a_2^4 B_{0,2,0}^{(\xi)} - a_2^2 B_{0,2,1}^{(\xi)}) p_{0,2}^{(0)} - a_1^2 B_{1,1,1}^{(\xi)} p_{2,0}^{(0)} = -\gamma^{-1} q_{0,2}^0$$

$$t_{0,0} p_{0,0}^{(0)} = P - t_{2,0} p_{2,0}^{(0)} - t_{0,2} p_{0,2}^{(0)}$$

$$\gamma^{-1}\delta = 2B_{0,0,0}^{(\xi)} p_{0,0}^{(0)} + a_1^2 B_{1,0,1}^{(\xi)} p_{2,0}^{(0)} + a_2^2 B_{0,1,1}^{(\xi)} p_{0,2}^{(0)}$$

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A SPECIAL RELATIONSHIP IN SPHEROIDAL WAVE FUNCTIONS AND ITS APPLICATION TO CONTACT PROBLEMS*

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A spectral and kindred relationship are set up by methods of the theory of the generalized potential /1/ for an integral operator generated by a symmetric difference kernel in the form of a Macdonald function in two identical semi-infinite intervals $\{(-\infty, -a), (a, \infty)\}$ that contain spheroidal wave functions. The formula for the expansion of an arbitrary function in these functions is also set up by a well-known method /2/. On the basis of the results obtained, a solution is then constructed for the integral equation of the contact problem of the impression of two identical stamps with half-plane bases into a half-space being deformed in a power-law form in the formulation of /3/.

This contact problem can be described by the same integral equation when the elastic modulus of a linearly elastic half-space changes with depth according to a power law /1/.

The spectral relationships in classical orthogonal polynomials for extensive classes of integral operators in mathematical physics are presented in /4,5/, where the method of orthogonal polynomials based on them is also elucidated, and numerous applications of this method are given to contact and mixed problems of elasticity theory. We also mention /6-9/ which are related directly to the investigation presented here.

1. Consider the integral equation

$$K\varphi_s = f_s(y), \quad K\varphi_s = \Big(\int_{-\infty}^{-a} + \int_a^{\infty}\Big) \frac{K_\mu(|s||y-\eta|)}{|y-\eta|^\mu}\varphi_s(\eta)\,d\eta, \quad |\mu| < \frac{1}{2} \tag{1.1}$$

in order to set up a spectral relationship for the integral operator $\psi_s(y) = K\varphi_s$, where $K_\mu(y)$ is the Macdonald function. To this end, following /8/, we introduce the function ($\Gamma(x)$ is the gamma function)

$$V(y, z, s) = U_s(y, z) = \int_{-\infty}^{\infty} U(x, y, z)\, e^{isx}\, dx = \tag{1.2}$$

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